National Examination — December 2014

04-BS-16 Discrete Mathematics

Duration: 3 hours

First Name: **Last Name:**

Examination Type: Closed book

Aids Permitted: Approved calculator and one aid sheet written on both sides of letter-size paper (8.5″ × 11″)

Notes:

☞ This exam paper contains 13 pages, including this cover page.

☞ There are 12 questions in this exam paper. Each question carries the same mark.
Answer **any 10** questions. If you attempt more than 10 questions, the best 10 marks will be taken.

☞ In case of doubt to any question, clearly state any assumptions made.

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10.	
11.	
12.	
Total	

1. [10 MARKS]

Part (a) [5 MARKS]

Verify that $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology.

Part (b) [5 MARKS]

For primitive statements p, q, determine if $(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$ is a tautology.

2. [10 MARKS]

Part (a) [2 MARKS]

For the universal set $\mathscr{U} = \{1, 2, 3, \ldots, 9, 10\}$, consider sets $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 4, 8\}$, $C = \{1, 2, 3, 5, 7\}$, and $D = \{2, 4, 6, 8\}$. Determine each of the following:

(1) $(A \cup B) \cap C$

(2) $\overline{(C \cap D)}$

Part (b) [4 MARKS]

Using the laws of set theory, simplify each of the following:

(1) $A \cap (B - A)$

(2) $(A \cap B) \cup (A \cap B \cap \overline{C} \cap D) \cup (\overline{A} \cap B)$

Part (c) [4 MARKS]

Using Venn diagrams or otherwise, for sets A, B, $C \subseteq \mathscr{U}$, investigate the truth or falsity of $A \triangle (B \cap C) = (A \triangle B) \cap (A \triangle C)$.

3. [10 MARKS]

Part (a) [5 MARKS]

Victoria tosses a fair coin seven times. Find the probability she gets four heads given that

(1) her first toss is a head;

(2) her first and last tosses are heads.

Part (b) [5 MARKS]

Victoria has a bag of 19 marbles of the same size. Nine of these marbles are red, six blue, and four white. She randomly selects three of the marbles, without replacement, from the bag. What is the probability that Victoria has withdrawn more red than white marbles?

4. [10 MARKS]

Part (a) [5 MARKS]

By the Principle of Mathematical Induction, establish $\sum_{i=1}^{n} i \cdot 2^i = 2 + (n-1) \cdot 2^{n+1}$, for all $n \geqslant 1$.

Part (b) [5 MARKS]

For $n \in \mathbf{Z}^+$, let $S(n)$ be the open statement $\sum_{i=1}^{n} i = \frac{\left(n + \frac{1}{2}\right)^2}{2}$.

(1) Show that the truth of $S(k)$ implies the truth of $S(k+1)$ for all $k \in \mathbf{Z}^+$.

(2) Is $S(n)$ true for all $n \in \mathbf{Z}^+$?

5. [10 MARKS]

Define the integer sequence $a_0, a_1, a_2, a_3, \ldots,$ recursively by

(1) $a_0 = 1,\ a_1 = 1,\ a_2 = 1;$ and

(2) for $n \geqslant 3,\ a_n = a_{n-1} + a_{n-3}.$

Show that $a_{n+2} \geqslant (\sqrt{2})^n$ for all $n \geqslant 0$.

6. [10 MARKS]

Part (a) [2 MARKS]

For each of the following functions, determine whether it is one-to-one and determine its range.

(1) $f : [-\pi/2, \pi/2] \to \mathbf{R}$, $f(x) = \sin x$.

(2) $f : [0, \pi] \to \mathbf{R}$, $f(x) = \sin x$.

Part (b) [2 MARKS]

Let $f : \mathbf{R} \to \mathbf{R}$ where $f(x) = x^2$. Determine $f(A)$ for the following subsets A taken from the domain $\mathbf{R}$.

(1) $A = [-7, 2]$

(2) $A = (-4, -3] \cup [5, 6]$

Part (c) [2 MARKS]

Determine each of the following:

(1) $\lfloor 2.3 - 1.6 \rfloor$

(2) $\lfloor 2.3 \rfloor - \lfloor 1.6 \rfloor$

Part (d) [4 MARKS]

Determine whether each of the following statements is true or false. If the statement is false, provide a counterexample.

(1) $\lfloor a \rfloor = \lceil a \rceil - 1$ for all $a \in \mathbf{R} - \mathbf{Z}$

(2) $-\lceil a \rceil = \lceil -a \rceil$ for all $a \in \mathbf{R}$

7. [10 MARKS]

Part (a) [4 MARKS]

Prove that if we select 101 integers from the set $S = \{1, 2, 3, \ldots, 200\}$, there exist m, n in the selection where $\gcd(m, n) = 1$.

Part (b) [6 MARKS] If eight distinct dice are rolled, what is the probability that all six numbers appear?

8. [10 MARKS]

Part (a) [4 MARKS] For each of the following functions $f : \mathbf{R} \to \mathbf{R}$, determine whether f is invertible, and, if so, determine f^{-1}.

(1) $f = \{(x, y) \mid y = x^3\}$

(2) $f = \{(x, y) \mid y = x^4 + x\}$

Part (b) [6 MARKS]

If 13 cards are dealt from a standard deck of 52, what is the probability that these 13 cards include at least one card from each suit?

9. [10 MARKS]

Part (a) [5 MARKS]

Solve the recurrence relation: $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ for $n \geqslant 0$, and $a_0 = 2$.

Part (b) [5 MARKS]

Solve the recurrence relation: $a_{n+1} - 2a_n = 2^n$, $n \geqslant 0$, $a_0 = 1$.

10. [10 MARKS]

Part (a) [5 MARKS]

(1) Sketch the loop-free undirected graphs with the following adjacency matrices

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

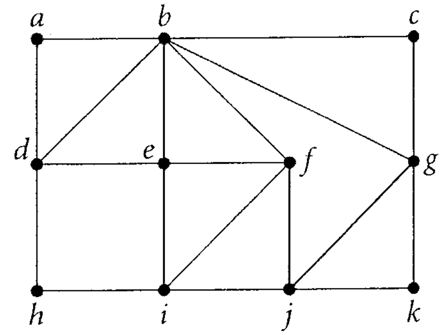
(2) Determine and justify whether or not these graphs are isomorphic.

Part (b) [5 MARKS]

There are 15 people at a party. Determine and justify if it possible for each of these people to shake hands with (exactly) three others.

11. [10 MARKS]

Consider the graph shown in the figure.



(1) Find an Euler circuit.

(2) If the edge $\{d, e\}$ is removed from this graph, find an Euler trail from d to e for the resulting subgraph.

12. [10 MARKS]

Part (a) [2 MARKS]

Determine the best "big-Oh" form for each of the following functions $f : \mathbf{Z}^+ \to \mathbf{R}$:

(1) $f(n) = 5n^2 + 3n\log_2 n$

(2) $f(n) = 2 + 4 + 6 + \cdots + 2n$

Part (b) [8 MARKS]

Determine $\gcd(1369, 2597)$ and express it as a linear combination of 1369 and 2597, i.e., determine $a, b \in \mathbf{Z}$ such that $\gcd(1369, 2597) = a \times 1369 + b \times 2597$.